EXTENDABLE CODIMENSION TWO SUBVARIETIES IN A GENERAL HYPERSURFACE

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ABSTRACT. We exhibit a class of *extendable* codimension 2 subvarieties in a general hypersurface of dimension at least 3 in projective space. As a consequence, we prove that a general hypersurface of degree d does not support globally generated indecomposable ACM bundles of any rank if their first Chern class $e \ll d$.

1. INTRODUCTION

Let Y be a smooth projective variety and $X \subset Y$ be a smooth subvariety. Relating the geometry of X and Y has been a long standing theme in algebraic geometry. Results in this context are usually referred to as Lefschetz theorems. The best known results are the *Grothendieck-Lefschetz* and *Noether-Lefschetz* theorems. A special case of the Noether-Lefschetz theorem says that for a very general hypersurface $X \subset \mathbb{P}^3$ of degree $d \geqslant 4$, any curve $C \subset X$ is a complete intersection in $\mathbb{P}^3$. In particular, $C = X \cap S$ for a surface $S \subset \mathbb{P}^3$ and thus *extendable* (there is a related notion of extendability in the literature, a very nice survey on which can be found in [Lop23]. See the references therein, especially [Wah87] and [BM87]).

More generally, throughout this article, we will say a codimension k subscheme $Z \subset X$ of a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ is *extendable* if $Z = X \cap \Sigma$ where $\Sigma \subset \mathbb{P}^{n+1}$ is a codimension k subscheme.

With a view to finding a generalisation of the Noether-Lefschetz theorem, Griffiths and Harris in [GH85], asked whether any curve in a general hypersurface $X \subset \mathbb{P}^4$ of degree $d \geqslant 6$ is extendable. The main idea is that codimension 2 subvarieties in projective spaces are already more complicated (for instance, not all of them are defined by 2 homogeneous polynomials) and the expectation was that perhaps the codimension 2 geometry of general hypersurfaces are no more complicated, thus establishing a Lefschetz type result.

C. Voisin in [Voi88] showed the existence of curves in smooth hypersurfaces $X \subset \mathbb{P}^4$ which were not cut out by surfaces in $\mathbb{P}^4$. One of the fundamental differences in these two cases is the following. Consider the normal bundle sequence for the inclusions $C \subset X \subset \mathbb{P}^{n+1}$:

$$0 \longrightarrow N_{C/X} \longrightarrow N_{C/\mathbb{P}^{n+1}} \to \mathcal{O}_C(d) \longrightarrow 0.$$

For smooth hypersurfaces in $\mathbb{P}^3$, this sequence splits if and only if C is extendable and hence a complete intersection (see [GH83]). However, this is no longer true once C is a curve in a smooth hypersurface $X \subset \mathbb{P}^4$. In this case, the splitting of the above sequence only implies that C is *infinitesimally extendable*, i.e., there exists a curve $D \subset X_{(1)}$ where $X_{(1)}$ is the first order thickening of X in $\mathbb{P}^4$ such that $C = D \cap X$. If $C \subset X$ (or more generally a codimension 2 subvariety Z in a smooth hypersurface of dimension $n \geqslant 4$) is, in addition, *arithmetically Cohen Macaulay* (henceforth, we abbreviate this as *ACM*), then it was shown in [MKRR09] that if C extends infinitesimally, then it is in fact extendable in the above sense. This fact was

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used to show the existence of a large class of counterexamples generalising Voisin's examples in [Voi88]. There are also examples of non-extendable subvarieties in higher codimension (see for instance [IN02]).

Coming back to the case of curves in hypersurfaces in $\mathbb{P}^4$, and their extendability, a conjecture in [RT19] proposes that any ACM curve C in a general hypersurface $X \subset \mathbb{P}^4$ of degree $d \geqslant 6$ is extendable if the number of generators of the canonical module of the curve C is less than or equal to 2. When the canonical module has a single generator, the curve C is *subcanonical* and the main result of [Rav09] (see also [MKRR07]) states that C is in fact a complete intersection. When the number of generators of the canonical module is 2, barring a few exceptions, this conjecture was settled in [RT22].

Extendability of codimension 2 ACM subvarieties in smooth hypersurfaces is related to a conjecture of Buchweitz-Greuel-Schreyer ([BGS87]) on the non-existence of low rank indecomposable ACM vector bundles and a generalisation of this conjecture (see [Fae13] and [RT19]), results on which are proven, for example, in [Tri16, Tri17, RT19, RT22]. It is also related to the *Ulrich complexity* of hypersurfaces ([Bea00, ES03]); we refer to [Cos17, Bea18, CMR+21] for an overview of this topic, see also [RT22, LR24a, LR24b].

In this article, we exhibit a bigger class of extendable curves C in a general hypersurface $X \subset \mathbb{P}^4$ of degree d. As a consequence, we prove a splitting result for ACM bundles E on X. The expert will immediately see that the results in this article are far from being sharp. Indeed our aim here has been to showcase how well-known and beautiful results available in the literature can be brought together to answer some rather long standing questions of interest.

Conventions. We work over the field of complex numbers $\mathbb{C}$. A *variety* is an integral separated scheme of finite type over $\mathbb{C}$. A *curve* (resp. *surface*) is a variety of dimension one (resp. two).

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2. Statements of the main results

In this section, we provide the statements of our main results. The aim of this article is to prove the following:

Theorem 1. *Let $X \subset \mathbb{P}^{n+1}$ be a general hypersurface of dimension $n \geqslant 3$ and degree d. A smooth ACM codimension 2 subvariety $Z \subset X$ is extendable if there exists a positive integer e such that*

(i) $\binom{e+5}{4} \leqslant 2d - 4$,
(ii) $I_{Z/X}(e)$ *is globally generated, and*
(iii) $\omega_Z \otimes \omega_X^{-1}(-e)$ *is globally generated.*

Here's an example of a situation in which such curves arise. Let E be a globally generated ACM bundle of rank r on a smooth, degree d hypersurface $X \subset \mathbb{P}^{n+1}$ with $n \geqslant 3$. Any choice of $r-1$ general sections yields an exact sequence

$$0 \longrightarrow \mathcal{O}_X^{\oplus r-1} \longrightarrow E \longrightarrow I_{Z/X}(e) \longrightarrow 0.$$

Here $Z \subset X$ is the codimension 2 subvariety defined by the vanishing of these $r-1$ sections, $I_{Z/X}$ is its ideal sheaf and e is the first Chern class of E. If e satisfies the inequality (i) in Theorem 1, then Z is extendable; i.e., $Z = X \cap \Sigma$ for some pure codimension 2 subscheme $\Sigma \subset \mathbb{P}^{n+1}$.

The proof of the above statement is based on an induction argument, the main step of which is proving the assertion when $n = 3$. The main ingredient of the proof in this case is the following:

Theorem 2. *Let X be a general hypersurface in $\mathbb{P}^4$ of degree d and let $C \subset X$ be an ACM curve. Consider the normal bundle sequence*

$$0 \longrightarrow N_{C/X} \longrightarrow N_{C/\mathbb{P}^4} \longrightarrow \mathcal{O}_C(d) \longrightarrow 0. \tag{1}$$

If C satisfies the following conditions:

(i) $\binom{e+5}{4} \leqslant 2d - 4$,

(ii) *there exists a smooth surface $S \in |I_{C/X}(e)|$, and*

(iii) $\omega_C \otimes \omega_X^{-1}(-e)$ *is globally generated,*

then the normal bundle sequence (1) *splits.*

When X is a general hypersurface in $\mathbb{P}^4$ of degree d and $C \subset X$ is an ACM curve, we show that the splitting of (1) implies extendability of C. Similar ideas were used by Voisin in [Voi92] in the context of the extendability of curves in K3 surfaces (in the sense discussed in [Lop23]).

As mentioned in the Introduction, extendability of pure codimension 2 subvarieties is intimately related with the splitting of ACM vector bundles (cf. Lemma 3). In this direction, we deduce the following by-product of our results:

Theorem 3. *Fix a positive integer e. Then a general hypersurface of dimension $n \geqslant 3$ and degree d satisfying the inequality $\binom{e+5}{4} \leqslant 2d-4$ does not support a non-split globally generated ACM bundle with first Chern class $c_1(E) = \mathcal{O}_X(e)$.*

Our proof of Theorem 2 makes use of the Beauville-Mérindol criterion (see [BM87]) for splitting of short exact sequences, combining it with Green's exactness criterion for Koszul complexes (see [Gre88]).

3. Preliminaries on Hartshorne-Serre correspondence

We recall the Hartshorne-Serre correspondence for codimension 2 subschemes in a smooth variety that will be crucial for us the sequel:

Theorem 4 ([Arr07, Theorem 1])**.** *Let X be a smooth, projective variety and $Z \subset X$ be a locally complete intersection subvariety of codimension 2. Let L be a line bundle such that*

(i) $H^2(X, L^{-1}) = 0$, *and*

(ii) $\omega_Z \otimes (\omega_X \otimes L)^{-1}$ *is globally generated by $(r-1)$ sections.*

Then there exists a rank r vector bundle E and an exact sequence

$$0 \longrightarrow \mathcal{O}_X^{\oplus r-1} \longrightarrow E \longrightarrow I_{Z/X} \otimes L \longrightarrow 0.$$

Furthermore, if $H^1(X, L^{-1}) = 0$, then E is unique up to an unique isomorphism.

Observe that when $n \geqslant 3$, $X \subset \mathbb{P}^{n+1}$ is a smooth hypersurface and $L = \mathcal{O}_X(e)$, we have

$$\omega_Z \otimes (\omega_X \otimes L)^{-1} = \omega_Z(n + 2 - d - e)$$

by adjunction. Also note that by the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(-d) \longrightarrow I_{Z/\mathbb{P}^{n+1}} \longrightarrow I_{Z/X} \longrightarrow 0,$$

$Z \subset X$ is ACM if and only if $Z \subset \mathbb{P}^{n+1}$ is ACM, a fact that we will frequently use often without any further reference. Let us record the following

Proposition 1. *Let* $X \subset \mathbb{P}^{n+1}$ *be a general hypersurface of degree* d *and dimension* $n \geqslant 3$. *Let* $e \geqslant 1$ *and let* $Z \subset X$ *be an ACM local complete intersection subvariety for which* $\omega_Z \otimes \omega_X^{-1}(-e)$ *is globally generated by* $(r-1)$ *sections. Then the associated vector bundle* E *(coming from Theorem 4) sitting in the exact sequence*

$$0 \longrightarrow \mathcal{O}_X^{\oplus r-1} \longrightarrow E \to I_{Z/X}(e) \longrightarrow 0. \tag{2}$$

is ACM. Moreover, E *is globally generated if and only if* $I_{Z/X}(e)$ *is globally generated.*

Proof. Taking dual of (2) gives rise to the 4-term exact sequence

$$0 \longrightarrow \mathcal{O}_X(-e) \longrightarrow E^\vee \longrightarrow \mathcal{O}_X^{\oplus r-1} \longrightarrow \mathcal{E}xt^1_X(I_{Z/X}(e), \mathcal{O}_X) \longrightarrow 0. \tag{3}$$

One has the identification $\mathcal{E}xt^1_X(I_{Z/X}, \omega_X) \cong \omega_Z$ using which (3) may be rewritten as

$$0 \longrightarrow \mathcal{O}_X(-e) \longrightarrow E^\vee \longrightarrow \mathcal{O}_X^{\oplus r-1} \longrightarrow \ell \longrightarrow 0 \tag{4}$$

where $\ell := \omega_Z \otimes \omega_X^{-1}(-e) = \omega_Z(n+2-d-e)$. Also, it follows from the construction in [Arr07] that

$$H^0(X, \mathcal{O}_X(a)^{\oplus r-1}) \longrightarrow H^0(C, \ell(a)) \text{ surjects for all } a \in \mathbb{Z} \tag{5}$$

where the map above is induced by the map $\mathcal{O}_X^{\oplus r-1} \longrightarrow \ell$ in (4).

Let E_1 be the torsion-free sheaf defined as the cokernel of the injection $\mathcal{O}_X(-e) \longrightarrow E^\vee$ in (4). Breaking up the sequence (4), we obtain the two short exact sequences

$$0 \longrightarrow \mathcal{O}_X(-e) \longrightarrow E^\vee \longrightarrow E_1 \longrightarrow 0, \tag{6}$$

$$0 \longrightarrow E_1 \longrightarrow \mathcal{O}_X^{\oplus r-1} \longrightarrow \ell \longrightarrow 0. \tag{7}$$

To this end, recall that $H^i_*(X, \mathcal{O}_X) = 0$ for $1 \leqslant i \leqslant n-1$. Passing to the cohomology of (7), we conclude that $H^1_*(X, E_1) = 0$ by (5). Consequently $H^1_*(X, E^\vee) = 0$ by (6) which by duality implies $H^{n-1}_*(X, E) = 0$. It follows that E is ACM since $H^i_*(X, E) = 0$ for $1 \leqslant i \leqslant n-2$ by (2). To see the second assertion, consider the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(X, \mathcal{O}_X^{\oplus r-1}) \otimes \mathcal{O}_X & \longrightarrow & H^0(X, E) \otimes \mathcal{O}_X & \longrightarrow & H^0(X, I_{Z/X}(e)) \otimes \mathcal{O}_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X^{\oplus r-1} & \longrightarrow & E & \longrightarrow & I_{Z/X}(e) & \longrightarrow & 0.
\end{array}$$

Since the left vertical map is surjective, it follows that the middle one is surjective if and only if the right one is so, whence the conclusion follows. □

As an useful consequence, we deduce the following:

Corollary 1. *Let the hypotheses be as in Proposition 1. Then the multiplication map*

$$H^0(Z, \ell(a)) \otimes H^0(Z, \mathcal{O}_Z(b)) \longrightarrow H^0(Z, \ell(a+b))$$

is surjective whenever $a, b \geqslant 0$.

Proof. Thanks to (5) (and the fact that $H^0(X, \mathcal{O}_X(m)) \longrightarrow H^0(Z, \mathcal{O}_Z(m))$ is surjective for all m), it is enough to check that

$$H^0(Z, \mathcal{O}_Z(a)) \otimes H^0(Z, \mathcal{O}_Z(b)) \longrightarrow H^0(Z, \mathcal{O}_Z(a+b))$$

is surjective whenever $a, b \geqslant 0$. For this, we note that we have a commutative diagram:

$$\begin{array}{ccc} H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(a)) \otimes H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(b)) & \longrightarrow & H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(a+b)) \\ \downarrow & & \downarrow \\ H^0(Z, \mathcal{O}_Z(a)) \otimes H^0(Z, \mathcal{O}_Z(b)) & \longrightarrow & H^0(Z, \mathcal{O}_Z(a+b)) \end{array} \tag{8}$$

The horizontal map on the top row is a surjection, and the vertical maps are surjective since Z is ACM. It follows that the bottom horizontal map is also a surjection. □

4. Surjectivity via Green's theorem

We now proceed to prove the main technical result that is needed in the proof of Theorem 2. Throughout this section, $X \subset \mathbb{P}^4$ is a smooth hypersurface of degree d, and $C \subset X$ is an ACM local complete intersection curve. We also assume that there is a smooth surface $S \in |I_{C/X}(e)|$ (in particular $e \geqslant 1$) i.e., we have the inclusions

$$C \subset S \subset X \subset \mathbb{P}^4$$

and the corresponding normal bundle sequence

$$0 \longrightarrow N_{C/S} \longrightarrow N_{C/X} \longrightarrow N_{S/X|C} \longrightarrow 0. \tag{9}$$

Since $N_{S/X} \cong \mathcal{O}_S(e)$, taking determinants, we have the identification

$$N_{C/S} \cong \det N_{C/X} \otimes \mathcal{O}_C(-e) \cong \omega_C \otimes {\omega_S}^{-1} = \ell$$

whence the normal bundle sequence in (9) may be rewritten as

$$0 \longrightarrow \ell \longrightarrow N_{C/X} \longrightarrow \mathcal{O}_C(e) \longrightarrow 0. \tag{10}$$

Taking cohomology, we get the sequence

$$0 \longrightarrow H^0(C, \ell) \longrightarrow H^0(C, N_{C/X}) \xrightarrow{\alpha} H^0(C, \mathcal{O}_C(e)) \longrightarrow \cdots .$$

Setting $W := \text{Image}(\alpha)$, we have an exact sequence

$$0 \longrightarrow H^0(C, \ell) \longrightarrow H^0(C, N_{C/X}) \longrightarrow W \to 0.$$

More generally, twisting (10) with $\mathcal{O}_C(b)$ for any $b \in \mathbb{Z}$, we also have exact sequences

$$0 \longrightarrow H^0(C, \ell(b)) \longrightarrow H^0(C, N_{C/X}(b)) \longrightarrow W_{b+e} \longrightarrow 0, \tag{11}$$

where

$$W_{b+e} := \text{Image}\left[\, H^0(C, N_{C/X}(b)) \longrightarrow H^0(C, \mathcal{O}_C(b+e)) \,\right].$$

Evidently $W = W_e$ in the above notation.

Lemma 1. *The vector spaces W_{b+e} for $b > 0$ are base point free linear subsystems of the space of global sections $H^0(C, \mathcal{O}_C(b+e))$.*

Proof. We have commutative diagrams

$$\begin{array}{ccc} H^0(C, N_{C/X}(b)) \otimes \mathcal{O}_C & \twoheadrightarrow & W_{b+e} \otimes \mathcal{O}_C \\ \downarrow & & \downarrow \\ N_{C/X}(b) & \twoheadrightarrow & \mathcal{O}_C(b+e) \end{array}$$

with surjective horizontal maps. By [Voi96] (see also [Pac04]), we see that $N_{C/X}(b)$ is globally generated for $b > 0$ and hence the left vertical arrow is surjective for $b > 0$. This implies that the right vertical map is also surective, i.e., W_{b+e} is a base point free linear subsystem of $H^0(C, \mathcal{O}_C(b+e))$ for $b > 0$. □

The main result of this section is the following:

Proposition 2. *Under the assumptions above, if* $\binom{e+5}{4} \leqslant 2d-4$, *then the multiplication map*

$$W_{d+e-5} \otimes H^0(C, \mathcal{O}_C(d)) \longrightarrow H^0(C, \mathcal{O}_C(2d+e-5))$$

is surjective.

Before we work out the proof of this result, we recall the following result of Green which will play a key role for us:

Theorem 5 ([Gre88, Theorem 2])**.** *Let* $\widetilde{W} \subset H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(a))$ *be a base-point free linear system. Then the Koszul complex*

$$\bigwedge^{p+1} \widetilde{W} \otimes H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k-a)) \longrightarrow \bigwedge^{p} \widetilde{W} \otimes H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) \longrightarrow \bigwedge^{p-1} \widetilde{W} \otimes H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k+a))$$

is exact in the middle provided that $\operatorname{codim}(\widetilde{W}) \leqslant k-p-a$.

Proof of Proposition 2. Let $\widetilde{W}$ denote the lift of W_{e+1} under the surjective map

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(e+1)) \twoheadrightarrow H^0(C, \mathcal{O}_C(e+1)).$$

We have a commutative square where the vertical maps are the multiplication maps, and the horizontal maps are the restriction maps:

$$\begin{array}{ccc} \widetilde{W} \otimes H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2d-6)) & \longrightarrow & W_{e+1} \otimes H^0(C, \mathcal{O}_C(2d-6) \\ \downarrow{\scriptstyle \widetilde{\mu}} & & \downarrow{\scriptstyle \mu} \\ H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2d+e-5)) & \twoheadrightarrow & H^0(C, \mathcal{O}_C(2d+e-5)) \end{array}$$

Claim 1. *Under the assumptions of Proposition 2, the map*

$$\mu : W_{e+1} \otimes H^0(C, \mathcal{O}_C(2d-6) \longrightarrow H^0(C, \mathcal{O}_C(2d+e-5))$$

is surjective.

Proof. In Green's result (Theorem 5 above), letting $p = 0$, $k = 2d+e-5$, and $a = e+1$, we see that if

$$\operatorname{codim}(W_{e+1}) \;=\; \operatorname{codim}(\widetilde{W}) \;\leqslant\; 2d-6, \tag{12}$$

then the left vertical map $\widetilde{\mu}$ is surjective which implies that the right vertical map μ is surjective as well. To prove (12), note that W_{e+1} is a base point free subspace of $H^0(C, \mathcal{O}_C(e+1))$ by Lemma 1. As the restriction map

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(e+1)) \longrightarrow H^0(C, \mathcal{O}_C(e+1))$$

is a surjection, it follows that

$$\operatorname{codim}(W_{e+1}) \leqslant h^0(\mathcal{O}_{\mathbb{P}^4}(e+1)) - 2 = \binom{e+5}{4} - 2.$$

Thus, (12) holds since $\binom{e+5}{4} \leqslant 2d-4$, whence the multiplication map μ is surjective. □

We continue with the proof of Proposition 2. Notice that $d > 6$ by hypothesis as $e \geqslant 1$. To finish the proof, we note that (11) induces the following commutative diagram

$$\begin{array}{ccc} H^0(C, \ell(1)) \otimes H^0(C, \mathcal{O}_C(1))^{\otimes(d-6)} & \hookrightarrow & H^0(C, N_{C/X}(1)) \otimes H^0(C, \mathcal{O}_C(1))^{\otimes(d-6)} \\ \downarrow & & \downarrow \\ H^0(C, \ell(d-5)) & \hookrightarrow & H^0(C, N_{C/X}(d-5)) \end{array}$$

which induces

$$\beta : W_{e+1} \otimes H^0(C, \mathcal{O}_C(1))^{\otimes(d-6)} \longrightarrow W_{d+e-5}$$

as the map between the cokernels of the horizontal maps in the above diagram. This map in turn gives rise to the commutative diagram

$$\begin{array}{ccc} W_{e+1} \otimes H^0(C, \mathcal{O}_C(1))^{\otimes(2d-6)} & \twoheadrightarrow & W_{e+1} \otimes H^0(C, \mathcal{O}_C(2d-6)) \\ {\scriptstyle \mu_1}\downarrow & & \downarrow{\scriptstyle \mu} \\ W_{d+e-5} \otimes H^0(C, \mathcal{O}_C(d)) & \xrightarrow{\mu_d} & H^0(C, \mathcal{O}_C(2d+e-5)). \end{array}$$

That the top horizontal map is surjective follows by a diagram similar to (8). The surjectivity of μ_d now follows by the surjectivity of μ proven in Claim 1. □

5. Proof of Theorem 2 via the Beauville-Mérindol criterion

We recall a very elegant splitting criterion, due to Beauville and Mérindol (see [BM87, Lemme 1]) for a sequence of vector bundles on a curve to be split. Since the proof is very short, we include it to enhance the ease of reading.

Lemma 2 (The Beauville-Mérindol criterion)**.** *Let C be a smooth projective curve and*

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0 \tag{13}$$

be a short exact sequence of bundles. This sequence splits if

(i) $H^0(C, F) \longrightarrow H^0(C, G)$ *is surjective, and*
(ii) the cup product map

$$\cup : H^0(C, G) \otimes H^0(C, E^\vee \otimes \omega_C) \longrightarrow H^0(C, E^\vee \otimes G \otimes \omega_C)$$

is surjective.

Proof. We first note that the boundary map $H^0(C, G) \xrightarrow{\partial} H^1(C, E)$ yields the map

$$\partial : H^0(C, G) \otimes H^0(C, E^\vee \otimes \omega_C) \longrightarrow \mathbb{C}.$$

The short exact sequence (13) corresponds to an element $\eta \in \mathrm{Ext}^1(G, E) \cong H^1(C, G^\vee \otimes E)$, and via Serre duality, we treat the element η as a map

$$\eta : H^0(C, G \otimes E^\vee \otimes \omega_C) \longrightarrow \mathbb{C}.$$

To this end, we note the following commutative diagram

$$\begin{array}{ccc} H^0(C, G) \otimes H^0(C, E^\vee \otimes \omega_C) & \xrightarrow{\partial} & \mathbb{C} \\ {\scriptstyle \cup}\downarrow & \nearrow{\scriptstyle \eta} & \\ H^0(C, G \otimes E^\vee \otimes \omega_C) & & \end{array} \tag{14}$$

Consequently, we have that $\partial = \eta \circ \cup$. Since the cup product map $\cup$ is surjective, we have $\eta = 0 \iff \partial = 0$, and the latter is zero by assumption. □

We will now apply the above to the normal bundle sequence (1) to prove Theorem 2:

Proof of Theorem 2. Note that the normal bundle $N_{C/X}$ is rank 2 bundle and as such we have

$$N^{\vee}_{C/X} \cong N_{C/X} \otimes \left(\det N_{C/X}\right)^{-1} \cong N_{C/X} \otimes \omega_X \otimes \omega_C^{-1}.$$

Consequently,

$$N^{\vee}_{C/X} \otimes \omega_C \cong N_{C/X} \otimes \omega_X. \tag{15}$$

By the Beauville-Mérindol criterion, we need to check that

(a) the map $\alpha : H^0(C, N_{C/\mathbb{P}^4}) \longrightarrow H^0(C, \mathcal{O}_C(d))$ is surjective, and
(b) the cup product map
$$H^0(C, \mathcal{O}_C(d)) \otimes H^0(C, N^{\vee}_{C/X} \otimes \omega_C) \longrightarrow H^0(C, N^{\vee}_{C/X} \otimes \omega_C(d))$$
is surjective.

Since X is a *general* hypersurface of degree d in $\mathbb{P}^4$, we have (see, for example, [BMK13, Proposition 3.2])

$$\text{Image}\left[H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d) \longrightarrow H^0(C, \mathcal{O}_C(d))\right] \subset \text{Image}\left[H^0(\mathbb{P}^4, N_{C/\mathbb{P}^4}(d) \longrightarrow H^0(C, \mathcal{O}_C(d))\right].$$

Recall that C is ACM, whence the map

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d)) \longrightarrow H^0(C, \mathcal{O}_C(d))$$

is surjective, which verifies condition (a).

For (b), using the identification in (15), we are reduced to proving that the cup product map

$$H^0(C, \mathcal{O}_C(d)) \otimes H^0(C, N_{C/X}(d-5)) \longrightarrow H^0(C, N_{C/X}(2d-5))$$

is surjective. Let us define

$$V_d := H^0(C, \mathcal{O}_C(d)).$$

Note that our hypotheses guarantee the existence of a short exact sequence as in (1) and the normal bundle sequence (10) for the inclusions $C \subset S \subset X$. So we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(C, \ell(d-5)) \otimes V_d & \longrightarrow & H^0(C, N_{C/X}(d-5)) \otimes V_d & \longrightarrow & W_{d+e-5} \otimes V_d & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(C, \ell(2d-5)) & \longrightarrow & H^0(C, N_{C/X}(2d-5)) & \longrightarrow & W_{2d+e-5} & \longrightarrow & 0
\end{array}$$

where the vertical maps are multiplication maps. Note that $H^0(C, \ell(d-5)) \neq 0$ since $d > 5$. Thus the first vertical map is surjective by Corollary 1. The rightmost vertical map is surjective by Proposition 2. By the snake lemma, it follows that the middle vertical map is also surjective. □

6. PROOFS OF THEOREM 1 AND THEOREM 3

Let us first make a note of the following elementary

Lemma 3. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d, and let $Z \subset X$ be a smooth codimension 2 subvariety defined by the exact sequence*

$$0 \longrightarrow \mathcal{O}_X^{\oplus r-1} \longrightarrow E \longrightarrow I_{Z/X}(e) \longrightarrow 0$$

where E is a bundle of rank r. If E is split, then Z is extendable.

Proof. Since E splits into a sum of line bundles, the map

$$\mathcal{O}_X^{\oplus r-1} \longrightarrow E \cong \bigoplus_{i=1}^{r} \mathcal{O}_X(a_i)$$

lifts to a map

$$\mathcal{O}_{\mathbb{P}^4}^{\oplus r-1} \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^4}(a_i).$$

The cokernel of this map is (a twist of) the ideal sheaf of a codimension 2 subscheme $\Sigma \subset \mathbb{P}^{n+1}$ which satisfies the condition that $Z = X \cap \Sigma$. This, in particular, implies that Σ doesn't have a divisorial component. Since $\mathrm{codim}_{\mathbb{P}^{n+1}}(\Sigma) \leqslant 2$ (see for e.g. [Ott95, Lemma 2.7]), we conclude that Σ is of pure codimension 2, whence Z is extendable. □

We are now ready to provide the proofs of Theorem 1 and Theorem 3:

Proof of Theorem 1. The proof is based by induction on the dimension n. Let us first deal with the base case:

Claim 2. *Theorem 1 holds when* $n = 3$.

Proof. Let $C := Z \subset X$ be a smooth ACM curve satisfying the hypotheses of Theorem 1. By Lemma 3, it is enough to show that E in (2) is split. Recall from [MKRR09, Section 2] that there exists a short exact sequence

$$0 \longrightarrow G \longrightarrow F \longrightarrow I_{C/X} \longrightarrow 0 \tag{16}$$

such that

(i) G is ACM and F is split,
(ii) $H^0_*(X, F) \longrightarrow H^0_*(X, I_{C/X})$ is surjective.

Now, since E is globally generated by Proposition 1, we may assume that there is a smooth $S \in |I_{C/X}(e)|$ by choosing a general r-dimensional subspace $V_r \subset H^0(E)$ containing

$$V_{r-1} := \text{Image}\left[H^0(X, \mathcal{O}_X^{\oplus r-1}) \longrightarrow H^0(E)\right]$$

(see for e.g. [Ott95, Teorema 2.8], also [CFK23, Remark 3.4]). Consequently, the normal bundle sequence for the inclusions $C \subset X \subset \mathbb{P}^4$ splits by Theorem 2, whence G splits by [MKRR09, Proposition 2]. Twisting the exact sequence (2) by $\mathcal{O}_X(-e)$, we obtain

$$0 \longrightarrow \mathcal{O}_X(-e)^{\oplus r-1} \longrightarrow E(-e) \longrightarrow I_{C/X} \longrightarrow 0. \tag{17}$$

Since F is split, we conclude that the map

$$H^0(X, F^\vee \otimes E(-e)) \longrightarrow H^0(X, F^\vee \otimes I_{C/X})$$

induced by the above exact sequence is surjective as $H^1(X, F^\vee \otimes \mathcal{O}_X(-e)) = 0$. Thus the map $F \longrightarrow I_{C/X}$ in (16) lifts to a map $F \longrightarrow E(-e)$ via (17). Consequently, defining

$$\widetilde{F} := F \oplus \mathcal{O}_X(-e)^{\oplus r-1}$$

and using snake lemma, we obtain the following diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathcal{O}_X(-e)^{\oplus r-1} & = & \mathcal{O}_X(-e)^{\oplus r-1} & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G & \longrightarrow & \widetilde{F} & \longrightarrow & E(-e) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G & \longrightarrow & F & \longrightarrow & I_{C/X} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{18}$$

Since G is split, we obtain $\mathrm{Ext}^1(E(-e), G) = 0$ (recall that E is ACM by Proposition 1). Consequently the middle row of (18) is split. Since $\widetilde{F}$ is split, E splits. □

Let us continue with the proof of Theorem 1. Now we carry out the induction step. Since the assertion is already proven for $n = 3$ in Claim 2, we assume $n \geqslant 4$. Recall the exact sequence

$$0 \longrightarrow \mathcal{O}_X^{\oplus r-1} \longrightarrow E \longrightarrow I_{Z/X}(e) \longrightarrow 0,$$

where E is a rank r globally generated ACM bundle on X (see Proposition 1). Setting $X_n := X$, $Z_{n-2} := Z$, and repeatedly restricting this sequence by general hyperplane sections X_i of dimension i, one obtains codimension 2 subvarieties Z_{i-2} of dimension $i - 2$, and the exact sequence

$$0 \longrightarrow \mathcal{O}_{X_i}^{\oplus r-1} \longrightarrow E_i \longrightarrow I_{Z_{i-2}/X_i}(e) \longrightarrow 0 \text{ for all } i \geqslant 3, \tag{19}$$

where $E_i := E|_{X_i}$. It is easy to verify that E_i is ACM for $i \geqslant 3$, whence $Z_{i-2} \subset X_i$ is ACM by (19) for i in the same range. As a result, the pair (Z_{i-2}, X_i) satisfies the hypotheses of the Theorem for all $i \geqslant 3$. Consequently, E_3 is split by the proof of Claim 2, and we inductively assume E_i is split for some $i < n$. Write $E_i \cong \bigoplus_{i=1}^r \mathcal{O}_{X_i}(a_i)$ and note that the composed map

$$E_{i+1} \longrightarrow E_i \cong \bigoplus_{i=1}^{r} \mathcal{O}_{X_i}(a_i)$$

lifts to a map

$$E_{i+1} \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}_{X_{i+1}}(a_i) \tag{20}$$

via the exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}_{X_{i+1}}(a_i - 1) \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}_{X_{i+1}}(a_i) \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}_{X_i}(a_i) \longrightarrow 0$$

as $H^1_*(X_{i+1}, E^\vee_{i+1}) = 0$ (recall that E_{i+1} is ACM). Since (20) is a map between vector bundles of the same rank, we conclude that it is an isomorphism. Indeed, this is a consequence of the fact that the determinant of the map is non-zero as it is so on X_i. This implies that E_{i+1} itself is a sum of line bundles, whence E is split by induction. Hence Z is extendable by Lemma 3. □

Proof of Theorem 3. By the argument of Theorem 1, it is enough to show that E_3 is split where $E_3 := E|_{X_3}$ and X_3 is a obtained by intersecting $n-3$ general hyperplane sections of X. As E_3 is globally generated with $c_1(E_3) = \mathcal{O}_{X_3}(e)$, a choice of $r-1$ general sections yields an exact sequence

$$0 \longrightarrow \mathcal{O}_{X_3}^{\oplus r-1} \longrightarrow E_3 \longrightarrow I_{Z_1/X_3}(e) \longrightarrow 0,$$

where I_{Z_1/X_3} is the ideal sheaf of a pure codimension 2 smooth ACM subscheme Z_1 in X_3. First assume $Z_1 = \emptyset$ whence $I_{Z_1/X_3} = \mathcal{O}_{X_3}$. In this case, clearly the above exact sequence is split as $H^1_*(X_3, \mathcal{O}_{X_3}) = 0$ whence E_3 is split. So, we may assume $Z_1 \neq \emptyset$, in particular $H^0(X_3, I_{Z_1/X_3}) = 0$. Since $Z_1 \subset X_3$ is ACM, we see that $H^0(Z_1, \mathcal{O}_{Z_1}) = 1$, in particular Z_1 is irreducible. To this end, applying the proof of Claim 2, we conclude that E_3 is split. □

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