A USEFUL HOMOLOGICAL RESULT

The purpose of this note is to provide details about the mapping cone argument used in the journal version of the proof of [LR24, Corollary 5.3]. Note that similar reasoning is used in various other articles, for e.g. [AK24, Proof of Proposition 3] and [TY22, Proof of Proposition 3.3]. We aim to prove the following:

Proposition 0.1. *Consider the commutative diagram*

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_2 & \xrightarrow{\alpha_2} & G_2 & \xrightarrow{\beta_2} & H_2 & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle h_2} & & \\
0 & \longrightarrow & F_0 \oplus F_1' & \xrightarrow{\alpha_1} & G_1' \oplus G_2 & \xrightarrow{\beta_1} & H_1 & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle h_1} & & \\
0 & \longrightarrow & F_0 & \xrightarrow{\alpha_0} & G_0 & \xrightarrow{\beta_0} & H_0 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{0.1}$$

with exact rows and exact right-most column. Assume that the middle column is a complex. Further assume:

- $g_2 : G_2 \to G_1' \oplus G_2$ *is the canonical injection, and*
- *the induced map* $f_1 : F_0 \oplus F_1' \to F_0$ *is the canonical projection.*

Then we have an exact complex:

$$0 \to F_2 \to F_1' \to G_1' \to G_0 \to 0.$$

We first prove three lemmas:

Lemma 0.2. *Consider the following commutative diagram*

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_2 & \xrightarrow{\alpha_2} & G_2 & \xrightarrow{\beta_2} & H_2 & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle h_2} & & \\
0 & \longrightarrow & F_1 & \xrightarrow{\alpha_1} & G_1 & \xrightarrow{\beta_1} & H_1 & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle h_1} & & \\
0 & \longrightarrow & F_0 & \xrightarrow{\alpha_0} & G_0 & \xrightarrow{\beta_0} & H_0 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{0.2}$$

whose rows are exact. Further assume:

$$\mathbf{H}: \quad 0 \to H_2 \xrightarrow{h_2} H_1 \xrightarrow{h_1} H_0 \to 0$$

is exact, and

$$\mathbf{G}: \quad 0 \to G_2 \xrightarrow{g_2} G_1 \xrightarrow{g_1} G_0 \to 0$$

is a complex. Then the diagram above can be completed by a complex

$$\mathbf{F}: \quad 0 \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \to 0$$

such that the mapping cone $\mathbf{M}(\alpha)$ *of* $\alpha : \mathbf{F} \to \mathbf{G}$ *given by*

$$\mathbf{M}(\alpha): \quad 0 \to F_2 \xrightarrow{\partial_3} F_1 \oplus G_2 \xrightarrow{\partial_2} F_0 \oplus G_1 \xrightarrow{\partial_1} G_0 \to 0$$

where the map $\partial_i : F_{i-1} \oplus G_i \to F_{i-2} \oplus G_{i-1}$ *is given by*

$$\partial_i(f, g) = (-f_{i-1}(f), \alpha_{i-1}(f) + g_i(g))$$

is exact.

Proof. The complex $\mathbf{F}$ is induced by the diagram above. Moreover, the long exact sequence

$$\cdots \to H_i(\mathbf{F}) \to H_i(\mathbf{G}) \to H_i(\mathbf{H}) \to \cdots$$

shows that

$$H_i(\mathbf{F}) \cong H_i(\mathbf{G}) \text{ for all } i.$$

Consequently, the long exact sequence

$$\cdots \to H_i(\mathbf{F}) \to H_i(\mathbf{G}) \to H_i(\mathbf{M}(\alpha)) \to \cdots$$

obtained from [Eis95, Proposition A3.19] proves the assertion. □

From now on, given $K_1 \oplus K_2$, we will denote the two projections by π_i. By abuse, we will not specify dependence of π_i on the given direct sum, which will be clear from the context.

Lemma 0.3. *In (0.2), assume* $G_1 = G_1' \oplus G_2$ *and the map* g_2 *is the canonical inclusion of the second factor. Then we have the complex:*

$$\mathbf{M}'(\alpha): \quad 0 \to F_2 \xrightarrow{\partial_3'} F_1 \xrightarrow{\partial_2'} F_0 \oplus G_1' \xrightarrow{\partial_1'} G_0 \to 0$$

where

- $\partial_3' = \pi_1 \circ \partial_3$,
- ∂_2' *is the composition (the first and the last are canonical injection and projection)*

$$F_1 \hookrightarrow F_1 \oplus G_2 \xrightarrow{\partial_2} F_0 \oplus G_1' \oplus G_2 \xrightarrow{\pi} F_0 \oplus G_1',$$

- ∂_1' *is the restriction of* ∂_1 *on its direct summand.*

Moreover $\mathbf{M}'(\alpha)$ *is exact.*

Proof. We first check that $\mathbf{M}'(\alpha)$ is a complex. Note that $g_1(g) = g_1(\pi_1(g), 0)$ for any $g \in G_1' \oplus G_2$.

Given $a \in F_2$:

$$\partial_2'(\partial_3'(a)) = \partial_2'(\pi_1(-f_2(a), \alpha_2(a)) = -\partial_2'(f_2(a)) = -\pi(\partial_2(f_2(a), 0)) = -\pi(-(f_1 f_2)(a), (\alpha_1 f_2)(a)) =$$
$$= -\pi(0, (g_2\alpha_2)(a)) = (0, -(\pi_1 g_2 \alpha_2)(a)) = (0, 0)$$

Given $a \in F_1$:

$$\partial_1'(\partial_2'(a)) = \partial_1'(\pi(\partial_2(a, 0)) = \partial_1'(\pi(-f_1(a), \alpha_1(a))) = \partial_1'(-f_1(a), (\pi_1\alpha_1)(a)) = \partial_1(-f_1(a), (\pi_1\alpha_1)(a), 0)$$
$$= -(\alpha_0 f_1)(a) + g_1((\pi_1\alpha_1)(a), 0) = -(g_1\alpha_1)(a) + g_1((\pi_1\alpha_1)(a), 0) = 0.$$

Now consider the diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & F_2 & = & F_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \partial_3} & & \downarrow{\scriptstyle \partial'_3} & & \\
0 & \longrightarrow & G_2 & \longrightarrow & F_1 \oplus G_2 & \longrightarrow & F_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \partial_2} & & \downarrow{\scriptstyle \partial'_2} & & \\
0 & \longrightarrow & G_2 & \longrightarrow & F_0 \oplus G'_1 \oplus G_2 & \xrightarrow{\pi} & F_0 \oplus G'_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle \partial'_1} & & \\
 & & 0 & \longrightarrow & G_0 & = & G_0 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where the horizontal maps are canonical injections or projections. Let us check commutativity.

Top right square is commutative by definition.

Middle left square is commutative as $\partial_2(0,g) = (0,0,g)$.

Middle right square is commutative as

$$(\pi\partial_2)(f,g) = \pi(-f_1(f), \alpha_1(f) + g_2(g)) = (-f_1(f), (\pi_1\alpha_1)(f)) = (\pi\partial_2)(f,0) = \partial'_2(f).$$

Bottom left square is commutative as $\partial_1(0,0,g) = 0$.

Bottom right square is commutative as for $f \in F_0$ and $g \in G'_1 \oplus G_2$:

$$\partial_1(f,g) = \alpha_0(f) + g_1(g) = \alpha_0(f) + g_1(\pi_1(g),0) = \partial_1(f,\pi_1(g),0) = \partial'_1(f,\pi_1(g)) = (\partial'_1\pi)(f,g).$$

Setting

$$\mathbf{G}_2 : \quad 0 \to G_2 \xrightarrow{\mathrm{Id}} G_2 \to 0,$$

we obtain the exact sequence

$$0 \to \mathbf{G}_2[-1] \to \mathbf{M}(\alpha) \to \mathbf{M}'(\alpha) \to 0$$

which gives the conclusion after passing to homology. □

Lemma 0.4. *Given a module M, suppose we have a complex:*

$$\mathbf{F}_2(M) : \quad 0 \to F_3 \xrightarrow{\epsilon_3} M \oplus F_2 \xrightarrow{\epsilon_2} M \oplus F_1 \xrightarrow{\epsilon_1} F_0 \to 0.$$

Assume

- $\epsilon_2(m,f) = (-m, (\pi_2 \circ \epsilon_2)(m,f))$.
- $\pi_1 \circ \epsilon_3 : F_3 \to M$ *is zero.*

Then we have the complex

$$\mathbf{F} : \quad 0 \to F_3 \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \to 0$$

where

- $f_3 = \pi_2 \circ \epsilon_3$,
- f_2 *is the composition*

$$F_2 \hookrightarrow M \oplus F_2 \xrightarrow{\epsilon_2} M \oplus F_1 \to F_1$$

where the maps are canonical injection and projection, and

- f_1 *is the composition of canonical injection and* ϵ_1*:*

$$F_1 \hookrightarrow M \oplus F_1 \xrightarrow{\epsilon_1} F_0.$$

Moreover, $H_i(\mathbf{F}) \cong H_i(\mathbf{F}_2(M))$ *for all* i.

Proof. Let us check that $\mathbf{F}$ is a complex.

First note that given $f \in F_3$:

$$(\epsilon_2\epsilon_3)(f) = \epsilon_2(0,(\pi_2\epsilon_3)(f)) = (0,(\pi_2\epsilon_2)(0,(\pi_2\epsilon_3)(f))) = (0,0)$$

whence

$$(\pi_2\epsilon_2)(0,(\pi_2\epsilon_3)(f)) = 0.$$

Now, we have

$$(f_2 f_3)(f) = f_2(\pi_2(\epsilon_3(f))) = (\pi_2\epsilon_2)(0,(\pi_2\epsilon_3)(f)) = \pi_2(0,(\pi_2\epsilon_2)(0,(\pi_2\epsilon_3)(f))) = (\pi_2\epsilon_2)(0,(\pi_2\epsilon_3)(f)) = 0$$

by the above.

Given $f \in F_2$:

$$(f_1 f_2)(f) = f_1((\pi_2\epsilon_2)(0,f)) = \epsilon_1(0,(\pi_2\epsilon_2)(0,f)) = (\epsilon_1\epsilon_2)(0,f) = 0.$$

Now consider the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & F_3 & = & F_3 & \longrightarrow & 0 & & \\
 & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle \epsilon_3} & & \downarrow & & \\
0 & \longrightarrow & F_2 & \longrightarrow & M \oplus F_2 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle \epsilon_2} & & \downarrow{\scriptstyle -\mathrm{Id}} & & \\
0 & \longrightarrow & F_1 & \longrightarrow & M \oplus F_1 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle \epsilon_1} & & \downarrow & & \\
0 & \longrightarrow & F_0 & = & F_0 & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where the horizontal maps are canonical injections or projections. Let us check commutativity.

Top left square is commutative as $\epsilon_3(f) = (0,(\pi_2\epsilon_3)(f))$.

Top right square is commutative as $(\pi_1 f_3)(f) = 0$.

Middle left square is commutative as

$$(0, f_2(f)) = (0,(\pi_2\epsilon_2)(0,f)) = (0,\pi_2(0,(\pi_2\epsilon_2)(0,f)) = (0,(\pi_2\epsilon_2)(0,f)) = \epsilon_2(0,f).$$

Middle right square is commutative by the property of ϵ_2.

Bottom left square is commutative by the definition of f_1.

Now note that we have the exact sequence of complexes

$$0 \to \mathbf{F} \to \mathbf{F}_2(M) \to \mathbf{M}[-1] \to 0$$

where

$$\mathbf{M} : 0 \to M \xrightarrow{-\mathrm{Id}} M \to 0.$$

The assertion follows by passing to homology. □

Proof of Proposition 0.1. We apply Lemmas 0.2 and 0.3 on (0.1) to obtain the exact complex

$$0 \to F_2 \xrightarrow{\delta_3} F_0 \oplus F_1' \xrightarrow{\delta_2} F_0 \oplus G_1' \xrightarrow{\delta_1} 0.$$

Note that

$$\pi_1(\delta_3(f)) = -\pi_1(f_1(f)) = 0 \text{ and } \delta_2(a,b) = (-a,(\pi_1\alpha_1)(a,b)).$$

Thus the assumptions of Lemma 0.4 is satisfied whence the assertion follows by the same lemma. □

REFERENCES

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